

DETERMINANT DENSITY AND BIPERIODIC ALTERNATING LINKS

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ABSTRACT. Let \mathcal{L} be any infinite biperiodic alternating link. We show that for any sequence of finite links that Følner converges almost everywhere to \mathcal{L} , their determinant densities converge to the Mahler measure of the 2-variable characteristic polynomial of the toroidal dimer model on an associated biperiodic graph.

1. INTRODUCTION

The determinant of a knot is one of the oldest knot invariants that can be directly computed from a knot diagram. For any knot or link K ,

$$\det(K) = |\det(M + M^T)| = |H_1(\Sigma_2(K); \mathbb{Z})| = |\Delta_K(-1)| = |V_K(-1)|,$$

where M is any Seifert matrix of K , $\Sigma_2(K)$ is the 2-fold branched cover of K , $\Delta_K(t)$ is the Alexander polynomial and $V_K(t)$ is the Jones polynomial of K (see, e.g., [15]).

In [6], with Jessica Purcell, we studied the volume and determinant density of alternating hyperbolic links approaching the infinite square weave \mathcal{W} , the biperiodic alternating link shown in Figure 1(a). The *volume density* of a hyperbolic link K with crossing number $c(K)$ is defined as $\text{vol}(K)/c(K)$, and the *determinant density* of K is defined as $2\pi \log \det(K)/c(K)$. The volume density is known to be bounded by the volume of the regular ideal octahedron, $v_{\text{oct}} \approx 3.66386$, and the same upper bound is conjectured for the determinant density. With a suitable notion of convergence of diagrams, called here *Følner convergence almost everywhere*, $K_n \xrightarrow{F} \mathcal{W}$ as in Definition 2.3 below, we proved:

Theorem 1.1. [6] *Let K_n be any alternating hyperbolic link diagrams with no cycles of tangles such that $K_n \xrightarrow{F} \mathcal{W}$. Then K_n is both geometrically and diagrammatically maximal:*

$$\lim_{n \rightarrow \infty} \frac{\text{vol}(K_n)}{c(K_n)} = \lim_{n \rightarrow \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_{\text{oct}}.$$

We define the volume density of \mathcal{W} as $\text{vol}(L)/c(L)$, where L is the finite toroidally alternating \mathbb{Z}^2 -quotient link shown in Figure 1(b). Here, $c(L)$ is the crossing number of the reduced alternating projection of L on the torus, which is minimal by [1], and $\text{vol}(L) = \text{vol}(T^2 \times I - L)$ is the hyperbolic volume of its complement in the thickened torus $T^2 \times I$. In this case, $c(L) = 4$ and $\text{vol}(L) = 4v_{\text{oct}}$ (see [6]). Hence, as $K_n \xrightarrow{F} \mathcal{W}$, the volume densities of K_n converge to the volume density of \mathcal{W} , $\text{vol}(L)/c(L) = v_{\text{oct}}$.

However, v_{oct} initially appears as a mysterious limit of the determinant densities. In [6], we proved that this limit is the spanning tree entropy of the infinite square grid graph, which is the planar projection graph of \mathcal{W} . Just as in the case of volume density, there is an analogous toroidal invariant of \mathcal{W} that appears as the limit of the determinant density. This invariant is the entropy of the toroidal dimer model on an associated biperiodic graph.

In this paper, we extend this diagrammatic result for \mathcal{W} to *any* biperiodic alternating link \mathcal{L} . We show that using the same type of convergence of finite link diagrams as in Theorem 1.1, their determinant densities converge for any biperiodic alternating link \mathcal{L} . Moreover, we identify their limit as the Mahler measure of the 2-variable polynomial arising from the toroidal dimer model on a planar biperiodic graph. Following the definitions in the two subsections below, we present our main result in Theorem 2.4.

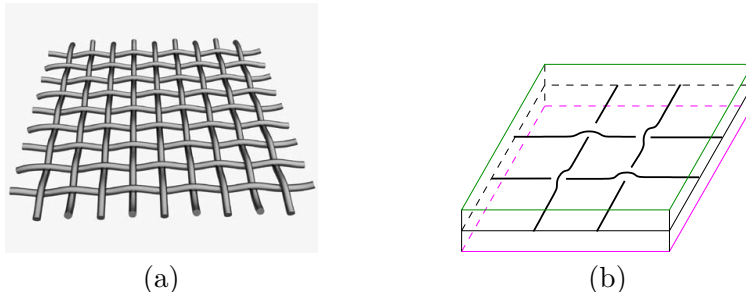


FIGURE 1. (a) Infinite weave \mathcal{W} . (b) Toroidally alternating quotient link L .

The main idea of the proof of Theorem 2.4 is to relate the limit of determinant densities of links approaching a biperiodic alternating link \mathcal{L} with the spanning tree entropy of a corresponding planar graph $G_{\mathcal{L}}$, and to relate this, in turn, to the entropy of the toroidal dimer model on a planar bipartite biperiodic graph $G_{\mathcal{L}}^b$. Thus, our main contribution is to bring some of the beautiful new results from the asymptotics of the toroidal dimer model to knot theory.

In Section 2 we give several required definitions and then state our main theorem. In Section 3, we discuss some aspects of the toroidal dimer model. In Section 4, we compute two examples that illustrate Theorem 2.4. Finally, in Section 5, we prove Theorem 2.4.

2. DEFINITIONS AND MAIN RESULT

2.1. Følner convergence of link diagrams. The following notion of convergence of graphs is well known, but the corresponding definition of convergence of link diagrams was introduced in [6]. We say an infinite graph G or an infinite alternating link \mathcal{L} is *biperiodic* if it is invariant under translations by a two-dimensional lattice Λ .

Definition 2.1. Let G be any infinite graph. For any finite subgraph G_n , the set ∂G_n is the set of vertices of G_n that share an edge with a vertex not in G_n . We let $|\cdot|$ denote the number of vertices in a graph. An exhaustive nested sequence of connected subgraphs, $\{G_n \subset G \mid G_n \subset G_{n+1}, \bigcup_n G_n = G\}$, is a *Følner sequence* for G if

$$\lim_{n \rightarrow \infty} \frac{|\partial G_n|}{|G_n|} = 0.$$

For a biperiodic planar graph G , we say $\{G_n \subset G\}$, is a *toroidal Følner sequence* for G if it is a Følner sequence for G such that $G_n \subset G \cap (n\Lambda)$.

Definition 2.2. Let G be any biperiodic planar graph. We will say that a sequence of planar graphs Γ_n *Følner converges almost everywhere* to G , denoted by $\Gamma_n \xrightarrow{F} G$, if

- (i) there are subgraphs $G_n \subset \Gamma_n$ that form a toroidal Følner sequence for G ,
- (ii) $\lim_{n \rightarrow \infty} |G_n|/|\Gamma_n| = 1$.

Definition 2.3. We will say that a sequence of alternating links K_n *Følner converges almost everywhere* to the biperiodic alternating link \mathcal{L} , denoted by $K_n \xrightarrow{F} \mathcal{L}$, if the respective projections graphs $\{G(K_n)\}$ and $G(\mathcal{L})$ satisfy $G(K_n) \xrightarrow{F} G(\mathcal{L})$; i.e., the following two conditions are satisfied:

- (i) there are subgraphs $G_n \subset G(K_n)$ that form a toroidal Følner sequence for $G(\mathcal{L})$,
- (ii) $\lim_{n \rightarrow \infty} |G_n|/c(K_n) = 1$.

For example, below is a Celtic knot diagram that could be in a sequence $K_n \xrightarrow{F} \mathcal{W}$:



As discussed in [6, 7], weaving knots $W(p, q)$ with $p, q \rightarrow \infty$ provide examples of infinite sequences that Følner converge almost everywhere to \mathcal{W} .

2.2. Mahler measure. The Mahler measure of a polynomial $p(z)$ is defined as

$$m(p(z)) = \frac{1}{2\pi i} \int_{S^1} \log |p(z)| \frac{dz}{z}$$

It is a natural measure of complexity of polynomials that is additive under multiplication.

By Jensen's formula, $m(p(z)) = \sum_{i=1}^n \max\{\log |\alpha_i|, 0\}$, where $\alpha_1, \dots, \alpha_n$ are roots of $p(z)$.

The Mahler measure of a two-variable polynomial $p(z, w)$ is defined similarly:

$$m(p(z, w)) = \frac{1}{(2\pi i)^2} \int_{S^1 \times S^1} \log |p(z, w)| \frac{dz}{z} \frac{dw}{w}.$$

Unlike the one-variable case, two-variable Mahler measures are much harder to compute and exact values of $m(p(z, w))$ are known only for certain families of two-variable polynomials.

Smyth's remarkable formula below provided the first evidence of a deep relationship between the Mahler measure of two-variable polynomials and hyperbolic volume. If K is the figure-8 knot, 4_1 , then $\text{vol}(K) = 2 v_{\text{tet}}$, where $v_{\text{tet}} \approx 1.01494$ is the hyperbolic volume of the regular ideal tetrahedron. Smyth [20] proved:

$$2\pi m(1 + x + y) = \frac{3\sqrt{3}}{2} L(\chi_{-3}, 2) = \text{vol}(K).$$

Later, Boyd and Rodriguez-Villegas [2] related the Mahler measure of A -polynomials of 1-cusped hyperbolic 3-manifolds to their hyperbolic volume. See the surveys [4, 21] on the Mahler measure of one and two variable polynomials.

2.3. Main result. For any finite link diagram K , let $G(K)$ denote the projection graph of K as above, and let G_K denote the Tait graph of K , which is the planar checkerboard graph for which a vertex is assigned to every shaded region and an edge to every crossing of K . Using the other checkerboard coloring yields the planar dual G_K^* . Thus, $e(G_K) = c(K)$. Any alternating link K is determined up to mirror image by its Tait graph G_K . Let $\tau(G)$ denote the number of spanning trees of G . By [11], for any connected reduced alternating link diagram,

$$\tau(G_K) = \det(K).$$

For a biperiodic alternating link \mathcal{L} , the projection graph $G(\mathcal{L})$ is biperiodic and can also be checkerboard colored. The two Tait graphs $G_{\mathcal{L}}$ and $G_{\mathcal{L}}^*$ are planar duals and are both biperiodic. We form the overlaid bipartite graph $G_{\mathcal{L}}^b = G_{\mathcal{L}} \cup G_{\mathcal{L}}^*$ as follows: The black vertices of G^b are the vertices of $G_{\mathcal{L}}$ and of $G_{\mathcal{L}}^*$; the white vertices of G^b are points of intersection of their edges. The overlaid graph $G_{\mathcal{L}}^b$ is a biperiodic balanced bipartite graph; i.e., the number of black vertices equals the number of white vertices in a fundamental domain. This makes it possible to define the toroidal dimer model on $G_{\mathcal{L}}^b$, and in Section 3 we explain how to obtain the characteristic polynomial $p(z, w)$ of the toroidal dimer model. The Λ -quotient link L is the toroidal link \mathcal{L}/Λ .

With these definitions, we can precisely state our main result:

Theorem 2.4. *Let \mathcal{L} be any biperiodic alternating link, with toroidally alternating Λ -quotient link L . Let $p(z, w)$ be the characteristic polynomial of the toroidal dimer model on $G_{\mathcal{L}}^b$. Then*

$$K_n \xrightarrow{F} \mathcal{L} \implies \lim_{n \rightarrow \infty} \frac{\log \det(K_n)}{c(K_n)} = \frac{m(p(z, w))}{c(L)}.$$

A similar limit for a particular closure of knots corresponding to sublattices of \mathcal{L} that grow in both directions is proved independently by Silver and Williams [19] using the Laplacian polynomial.

We will call the right-hand side of the above equation the determinant density of \mathcal{L} .

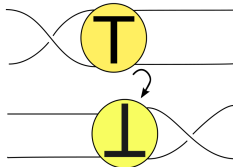


FIGURE 2. Flype move on a link diagram (figure from [23]).

A *flype*, shown in Figure 2, is a local move on link diagrams that has a rich history. Tait and Little started classifying alternating links using flypes, and they conjectured that two reduced alternating diagrams represent the same link if and only if they are related by flypes; i.e., one can be obtained from the other by a sequence of flypes. A century later, Menasco and Thistlethwaite [17] proved the “Tait Flying Conjecture” for all alternating links.

Corollary 2.5. *Let \mathcal{L} and \mathcal{L}' be any biperiodic alternating links, such that their toroidally alternating Λ -quotient links L and L' are related by flypes. Then the determinant densities of \mathcal{L} and \mathcal{L}' are equal.*

Proof. In the limit above, both $\det(K_n)$ and $c(K_n)$ are invariant under flypes. \square

3. TOROIDAL DIMER MODEL

The study of the dimer model is an active research area in statistical mechanics (see the excellent introductory lecture notes [9, 13]). A *dimer covering* (or *perfect matching*) of a graph is a subset of edges that cover every vertex exactly once; i.e., a pairing of adjacent vertices. The dimer model is the study of the set of dimer coverings of G . As we discuss below, it is also related to the spanning tree model of an associated planar graph.

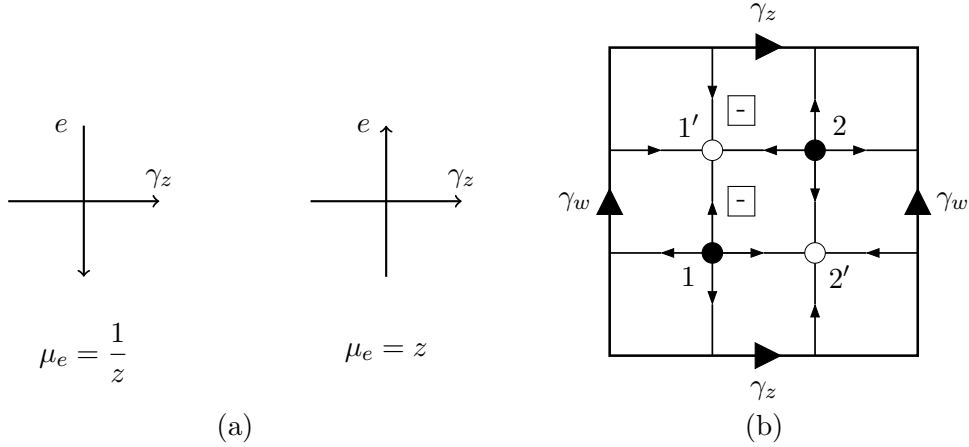


FIGURE 3. (a) Edge weights $\mu_e = z^{\gamma_z \cdot e}$ to compute $\kappa(z, w)$. (b) Toroidal bipartite graph G with a choice of Kasteleyn weighting.

Planar graphs. The simplest case is when G is a finite balanced bipartite planar graph, with edge weights μ_e for each edge e in G . A *Kasteleyn weighting* is a choice of sign for each edge, such that each face of G with 0 mod 4 edges has an odd number of signs, and each face with 2 mod 4 edges has an even number of signs. A *Kasteleyn matrix* κ is a weighted adjacency matrix of G , such that rows are indexed by black vertices, and columns by white vertices. The matrix coefficients are $\pm\mu_e$, with the sign determined by the Kasteleyn weighting. Then, taking the sum over all dimer coverings M of G , the *partition function* $Z(G)$ satisfies (see [9, 13]):

$$Z(G) := \sum_M \prod_{e \in M} \mu_e = |\det \kappa|.$$

With $\mu_e = 1$ for all edges, $Z(G)$ is the number of dimer coverings of G . Also see [10] for relations between dimer coverings of planar graphs and knot theory.

Toroidal graphs. Now, let G be a finite balanced bipartite toroidal graph. As in the planar case, we choose a Kasteleyn weighting on edges of G . We then choose oriented simple closed curves γ_z and γ_w on T^2 , transverse to G , representing a basis of $H_1(T^2)$. We orient each edge e of G from its black vertex to its white vertex. The weight on e is

$$\mu_e = z^{\gamma_z \cdot e} w^{\gamma_w \cdot e},$$

where \cdot denotes the signed intersection number of e with γ_z or γ_w . For example, see Figure 3. The Kasteleyn matrix $\kappa(z, w)$ is the weighted adjacency matrix with rows indexed by black vertices and columns by white vertices, and matrix entries $\pm\mu_e$, with the sign determined by the Kasteleyn weighting. The *characteristic polynomial* is defined as

$$p(z, w) = \det \kappa(z, w).$$

See Section 4 for examples. With μ_e as above, the number of dimer coverings of G is given by (see [9, 13]):

$$Z(G) = \frac{1}{2} | -p(1, 1) + p(-1, 1) + p(1, -1) + p(-1, -1) |.$$

Biperiodic graphs. Finally, let G be a biperiodic bipartite planar graph G , so that translations by a two-dimensional lattice Λ act by isomorphisms of G . Let G_n be the finite balanced bipartite toroidal graph given by the quotient $G/(n\Lambda)$. Kenyon, Okounkov and Sheffield [12] gave an explicit expression for the growth rate of the toroidal dimer model on $\{G_n\}$:

Theorem 3.1. [12, Theorem 3.5]

$$\log Z(G) := \lim_{n \rightarrow \infty} \frac{1}{n^2} \log Z(G_n) = m(p(z, w)).$$

The quantity $\log Z(G)$ on the left is called the entropy of the toroidal dimer model, or the partition function per fundamental domain. Theorem 3.1 says that, independent of any choice of Kasteleyn weighting and any choice of homology basis for the Λ -action, the entropy of any toroidal dimer model is given by the Mahler measure of its characteristic polynomial.

4. EXAMPLES

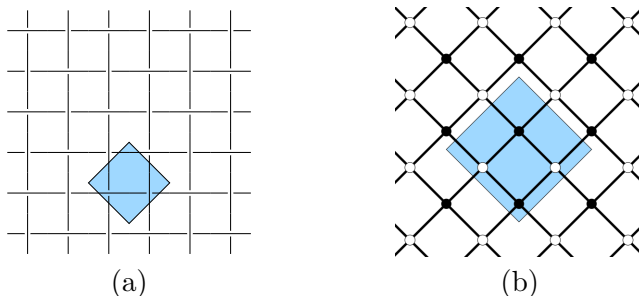


FIGURE 4. (a) Infinite square weave \mathcal{W} and fundamental domain for L .
(b) Bipartite graph $G_{\mathcal{W}}^b$ and fundamental domain for G_L^b .

4.1. Square weave. As mentioned in the introduction, Figure 1 shows the infinite square weave \mathcal{W} . Figure 4(a) shows a slightly different fundamental domain from Figure 1(b) and the toroidally alternating link L_1 with $c(L_1) = 2$. Both of the Tait graphs of \mathcal{W} are the infinite square grid. They are shown overlaid in Figure 4(b), which shows the biperiodic bipartite graph $G_{\mathcal{W}}^b$, as well as the fundamental domain for $G_{L_1}^b$, which matches the toroidal graph shown in Figure 3(b).

We now compute $p(z, w) = \det \kappa(z, w)$ for $G = G_{\mathcal{W}}^b$, as described in Section 3, and in more detail in [9, 13]. Using Figure 3(b) with the ordering as shown,

$$\kappa(z, w) = \begin{bmatrix} -1 - 1/z & 1 + w \\ 1 + 1/w & 1 + z \end{bmatrix}, \quad p(z, w) = - \left(4 + \frac{1}{w} + w + \frac{1}{z} + z \right)$$

Exact computations ([3, 22]) imply that $2\pi m(p(z, w)) = 2v_{\text{oct}}$. Therefore, by Theorems 1.1 and 2.4, v_{oct} is the limit of both determinant densities and volume densities for $K_n \xrightarrow{F} \mathcal{W}$:

$$\lim_{n \rightarrow \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi m(p(z, w))}{c(L_1)} = v_{\text{oct}} = \lim_{n \rightarrow \infty} \frac{\text{vol}(K_n)}{c(K_n)}.$$

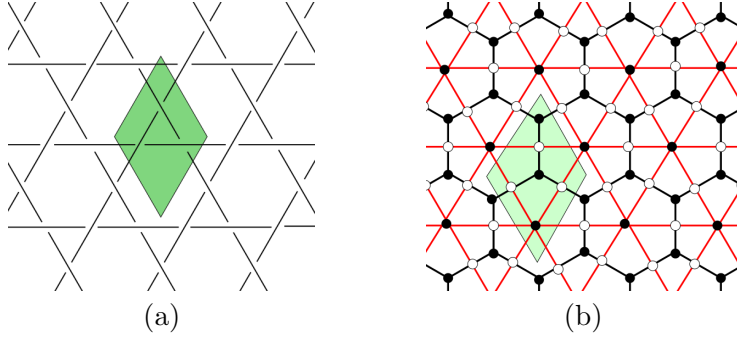


FIGURE 5. (a) Diagram of biperiodic triaxial link \mathcal{L} , and fundamental domain for L . (b) Bipartite graph $G_{\mathcal{L}}^b$ and fundamental domain for G_L^b .

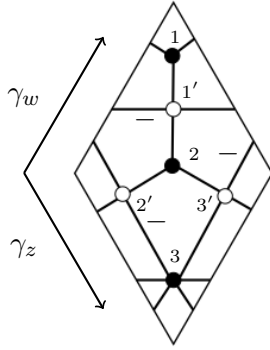


FIGURE 6. For the triaxial link \mathcal{L} , the toroidal graph G_L^b , with a choice of homology basis, ordered vertices and a choice of Kasteleyn weighting on edges.

4.2. Triaxial link. Figure 5(a) shows part of the biperiodic alternating diagram of the *triaxial link* \mathcal{L} , and the fundamental domain for the toroidally alternating link L . Its projection graph $G(\mathcal{L})$ is the trihexagonal tiling. The Tait graphs of \mathcal{L} are the regular hexagonal and triangular tilings, shown overlaid in Figure 5(b) to form the biperiodic balanced bipartite graph $G_{\mathcal{L}}^b$.

We now compute $p(z, w) = \det \kappa(z, w)$ for $G = G_{\mathcal{L}}^b$, as described in Section 3, and in more detail in [9, 13]. Using Figure 6, with the homology basis, ordered vertices and a choice of Kasteleyn weighting on edges as shown,

$$\kappa(z, w) = \begin{bmatrix} 1 & z & w \\ 1 & 1 & 1 \\ 1/z - 1/w & 1/w - 1 & 1 - 1/z \end{bmatrix}, \quad p(z, w) = 6 - \left(\frac{1}{w} + w + \frac{1}{z} + z + \frac{w}{z} + \frac{z}{w} \right)$$

Using exact computations ([3, 22]) we can verify that $2\pi m(p(z, w)) = 10v_{\text{tet}}$, where $v_{\text{tet}} \approx 1.01494$ is the hyperbolic volume of the regular ideal tetrahedron. Therefore, by Theorem 2.4,

$$\lim_{n \rightarrow \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \frac{2\pi m(p(z, w))}{c(L)} = \frac{10v_{\text{tet}}}{3}.$$

Moreover, in [8] we show that for the triaxial link \mathcal{L} ,

$$\lim_{n \rightarrow \infty} \frac{\text{vol}(K_n)}{c(K_n)} = \frac{\text{vol}(T^2 \times I - L)}{c(L)} = \frac{10v_{\text{tet}}}{3}.$$

Although for the square weave and the triaxial link, the volume and determinant densities both converge to the volume density of the toroidal link, this does not seem to be true in general. We discuss many such examples in [8].

5. PROOF OF THEOREM 2.4

Henceforth, let $G(\mathcal{L})$ and $G_{\mathcal{L}}$ be the projection graph and Tait graph of \mathcal{L} , respectively, both of which are Λ -biperiodic. Let K_n be alternating link diagrams that Følner converge almost everywhere to \mathcal{L} , so that $G(K_n) \xrightarrow{F} G(\mathcal{L})$. Let $G_n \subset G(K_n)$ form the toroidal Følner sequence for $G(\mathcal{L})$. Note that $|G(K_n)| = c(K_n)$.

Lemma 5.1. *As K_n Følner converges almost everywhere to \mathcal{L} , the sequence of Tait graphs G_{K_n} Følner converges almost everywhere to $G_{\mathcal{L}}$; i.e.,*

$$K_n \xrightarrow{F} \mathcal{L} \implies G_{K_n} \xrightarrow{F} G_{\mathcal{L}}$$

Proof. The choice of checkerboard coloring of faces of $G(\mathcal{L})$ to form $G_{\mathcal{L}}$ induces a checkerboard coloring of faces of $G(K_n)$, and hence a unique choice of Tait graphs G_{K_n} . We define $H_n \subset G_{K_n}$ as follows: The edge $e \in G_{K_n}$ is an edge of H_n if and only if the corresponding vertex $v \in G(K_n)$ is a vertex of $G_n - \partial G_n$. Because G_n is an exhaustive nested sequence of connected subgraphs of $G(\mathcal{L}) \cap (n\Lambda)$, it follows immediately that H_n is an exhaustive nested sequence of connected subgraphs of $G_{\mathcal{L}} \cap (n\Lambda)$.

Since \mathcal{L} is biperiodic, there is a positive integer d such that $\max(\deg(G(\mathcal{L})), \deg(G_{\mathcal{L}})) \leq d$. To prove the Følner condition, let $v \in \partial H_n$. As a vertex in $G_{\mathcal{L}}$, v is incident to a collection of edges $\{e_i \in H_n\}$ and $\{e'_j \notin H_n\}$, with $0 < i, j < d$. By definition of H_n , these edges correspond to vertices $\{v_i, v'_j \in G_n\}$, such that some $v'_j \in \partial G_n$. Hence, $|\partial H_n| \leq d |\partial G_n|$. Also by definition of H_n , $|G_n| \leq d |H_n|$. Therefore,

$$0 \leq \frac{|\partial H_n|}{|H_n|} \leq \frac{d |\partial G_n|}{\frac{1}{d} |G_n|} = d^2 \frac{|\partial G_n|}{|G_n|} \xrightarrow{n \rightarrow \infty} 0$$

Let v_n^{out} and e_n^{out} be the numbers of vertices and edges, respectively, of $G_{K_n} - H_n$. To prove item (ii) of Definition 2.2, we will show $\lim_{n \rightarrow \infty} v_n^{\text{out}}/|G_{K_n}| = 0$.

$$\begin{aligned} 0 \leq \frac{v_n^{\text{out}}}{|G_{K_n}|} &\leq \frac{e_n^{\text{out}}}{|G_{K_n}|} \leq \frac{|G(K_n) - (G_n - \partial G_n)|}{|H_n|} \leq \frac{|G(K_n) - (G_n - \partial G_n)|}{\frac{1}{d} |G_n|} \\ (1) \quad &\leq \frac{d(|G(K_n)| - |G_n| + |\partial G_n|)}{|G_n|} = \frac{d(c(K_n) - |G_n|)}{c(K_n)} \cdot \frac{c(K_n)}{|G_n|} + \frac{d |\partial G_n|}{|G_n|} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

The final limit follows from $\lim_{n \rightarrow \infty} |G_n|/c(K_n) = 1$ and the Følner condition. \square

Lemma 5.2. *As K_n Følner converges almost everywhere to \mathcal{L} , with L its Λ -quotient link,*

$$\lim_{n \rightarrow \infty} \frac{e(G_{K_n})}{|G_{K_n}|} = \frac{e(G_L)}{|G_L|}.$$

Proof. Let $H_n \subset G_{K_n}$ form the toroidal Følner sequence for $G_{\mathcal{L}}$ as in Lemma 5.1. Let $v_n^{\text{in}}, v_n^{\text{out}}, e_n^{\text{in}}, e_n^{\text{out}}$ be the numbers of vertices and edges, respectively, of H_n and $G_{K_n} - H_n$.

First, we claim that

$$(2) \quad \lim_{n \rightarrow \infty} \frac{v_n^{\text{out}}}{e_n^{\text{in}}} = 0.$$

To prove this claim, for every integer $k > 0$, let $|f_n^k|$ denote the number of k -faces of $G(K_n)$ that are not contained in G_n . Hence, $v_n^{\text{out}} = |G_{K_n} - H_n| \leq \sum_k |f_n^k|$. By counting vertices, $\sum_k k |f_n^k| \leq 4 |G(K_n) - G_n|$. The factor 4 appears because every vertex belongs to four faces, so it will be counted at most four times in the sum. Now, since $e_n^{\text{in}} = |G_n - \partial G_n|$, $|G(K_n)| = c(K_n)$, and using the limits in Definition 2.3, we have

$$0 \leq \frac{v_n^{\text{out}}}{e_n^{\text{in}}} \leq \frac{\sum_k |f_n^k|}{|G_n - \partial G_n|} \leq \frac{\sum_k k |f_n^k|}{|G_n - \partial G_n|} \leq \frac{4 |G(K_n) - G_n|}{|G_n - \partial G_n|} \xrightarrow{n \rightarrow \infty} 0.$$

We can now complete the proof of the lemma:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{e(G_{K_n})}{|G_{K_n}|} &= \lim_{n \rightarrow \infty} \frac{e_n^{\text{in}} + e_n^{\text{out}}}{|G_{K_n}|} = \lim_{n \rightarrow \infty} \frac{e_n^{\text{in}}}{|G_{K_n}|} \quad \text{by equation (1).} \\ \lim_{n \rightarrow \infty} \frac{|G_{K_n}|}{e_n^{\text{in}}} &= \lim_{n \rightarrow \infty} \frac{v_n^{\text{in}} + v_n^{\text{out}}}{e_n^{\text{in}}} = \lim_{n \rightarrow \infty} \frac{v_n^{\text{in}}}{e_n^{\text{in}}} \quad \text{by equation (2).} \end{aligned}$$

The biperiodicity of \mathcal{L} implies that the final limit is the corresponding ratio for the Λ -quotient link L . \square

Spanning trees and dimers. For any finite plane graph G , let G^b be the balanced bipartite graph as in Section 2: After overlaying G and G^* , the black vertices of G^b are the vertices of G and of G^* ; the white vertices are points of intersection of their edges. To make G^b balanced, we then delete the vertex of G^* corresponding to the unbounded face and a vertex of G adjacent to the unbounded face, along with all incident edges to these vertices (see Figure 7). Euler's formula implies that G^b is a balanced bipartite graph, and that $|G^b| = 2e(G)$.

By [5, 18], the spanning trees of G are in bijection with the dimer coverings of G^b ; i.e., $\tau(G) = Z(G^b)$. Hence, for any alternating link K , we have

$$(3) \quad \frac{\log \det(K)}{c(K)} = \frac{\log \tau(G_K)}{e(G_K)} = \frac{\log Z(G_K^b)}{c(K)}$$

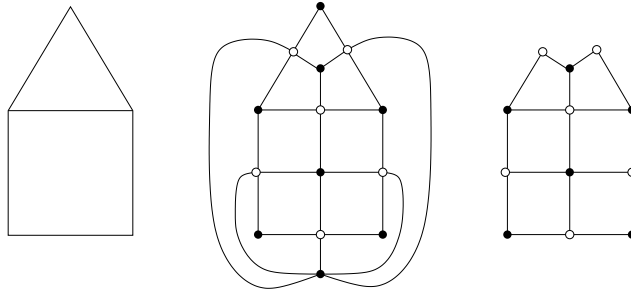


FIGURE 7. Graph G , overlaid graph $G \cup G^*$, balanced bipartite graph G^b

Because $G(\mathcal{L})$ is biperiodic and 4-valent, the overlaid graph $G_{\mathcal{L}}^b = G_{\mathcal{L}} \cup G_{\mathcal{L}}^*$ is already a biperiodic balanced bipartite graph. So we can consider the toroidal dimer model on $G_{\mathcal{L}}^b$.

We now relate the entropy of the toroidal dimer model on $G_{\mathcal{L}}^b$ with the spanning tree entropy of $G_{\mathcal{L}}$.

If $H_n \xrightarrow{F} G$, then the spanning tree entropy of G is defined as

$$h(G) = \lim_{n \rightarrow \infty} \frac{\log \tau(H_n)}{|H_n|}.$$

In [6], the spanning tree entropy of the infinite square grid was used to prove the determinant density limit in Theorem 1.1. For more details on spanning tree entropy and a broader context, see [16]. If G is a biperiodic planar graph, we define the *normalized spanning tree entropy* as

$$\tilde{h}(G) = \lim_{n \rightarrow \infty} \frac{\log \tau(H_n)}{n^2}.$$

Hence, if G is Λ -biperiodic and $H_n = G \cap (n\Lambda)$, then $\tilde{h}(G) = |H_1| h(G)$.

Proposition 5.3. *Let G be a Λ -biperiodic planar graph, and let $G^b = G \cup G^*$ be the overlaid graph, which is a bipartite biperiodic planar graph. Take the natural exhaustions of G^b by finite toroidal graphs $G_n^b = G^b / (n\Lambda)$, and of G by finite planar graphs $H_n = G \cap (n\Lambda)$. Then as $n \rightarrow \infty$, the entropy of the toroidal dimer model of G^b equals the normalized spanning tree entropy of G ; i.e.,*

$$\log Z(G^b) = \lim_{n \rightarrow \infty} \frac{\log Z(G_n^b)}{n^2} = \lim_{n \rightarrow \infty} \frac{\log \tau(H_n)}{n^2}$$

Proof. Temperley's bijection between spanning trees and dimers on the square grid was extended by Burton and Pemantle [5] to general planar graphs, and further extended by Kenyon, Propp and Wilson [14] to directed weighted planar graphs. Their main result is that there is a measure-preserving bijection between oriented weighted spanning trees of H_n and dimer coverings on G_n^b . Hence, the claim follows in principle from [14] which, although not stated for infinite planar graphs, holds for large planar graphs. The entropy in the infinite case can be obtained from a limit of large planar graphs.¹ \square

The following proposition shows that spanning tree entropy is not affected by peripheral changes in the graphs that converge as we defined above.

Definition 5.4. [16, p. 498] Given $R > 0$ and a finite graph G , let $\mathcal{E}_R(G)$ be the distribution of the number of edges in the ball of radius R about a random vertex of G . The sequence of graphs G_n is *tight* if for each R , the sequence of corresponding distributions $\mathcal{E}_R(G_n)$ is tight. In other words, for each $R > 0$,

$$\limsup_{t \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(\mathcal{E}_R(G_n) > t) = 0.$$

Proposition 5.5. [16, Corollary 3.8] *Let G_n be any tight sequence of finite connected graphs with bounded average degree such that $\lim_{n \rightarrow \infty} \frac{\log \tau(G_n)}{|G_n|} = h$. If H_n is a sequence of connected subgraphs of G_n such that*

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\#\{x \in V(H_n) : \deg_{H_n}(x) = \deg_{G_n}(x)\}}{|G_n|} = 1,$$

then $\lim_{n \rightarrow \infty} \frac{\log \tau(H_n)}{|H_n|} = h$.

¹We thank Richard Kenyon for this observation.

Proof of Theorem 2.4. By Lemma 5.1, $K_n \xrightarrow{F} \mathcal{L} \implies G_{K_n} \xrightarrow{F} G_{\mathcal{L}}$. Let $H_n \subset G_{K_n}$ form the toroidal Følner sequence for $G_{\mathcal{L}}$, with the planar graphs $H_n \subset G_{\mathcal{L}} \cap (n\Lambda)$. However, since H_n is also exhaustive, we may assume $H_n = G_{\mathcal{L}} \cap (n\Lambda)$; Proposition 5.5 implies that the spanning tree entropies are equal in either case. The biperiodicity of \mathcal{L} implies

$$(5) \quad \lim_{n \rightarrow \infty} \frac{|H_n|}{n^2 |G_L|} = 1.$$

In order to apply Proposition 5.5 for the graphs $H_n \subset G_{K_n}$, we now verify the required conditions. Lemma 5.2 implies that G_{K_n} have bounded average degree. Since $H_n \subset G_{\mathcal{L}}$ which is biperiodic, for any $R > 0$ there exists sufficiently large $t > 0$ such that $\mathbb{P}(\mathcal{E}_R(H_n) > t) = 0$. Hence, the conditions in Definition 2.2 applied to $H_n \subset G_{K_n} \xrightarrow{F} G_{\mathcal{L}}$ imply that for any $R > 0$ and sufficiently large $t > 0$,

$$0 \leq \mathbb{P}(\mathcal{E}_R(G_{K_n}) > t) \leq \frac{|G_{K_n} - H_n|}{|G_{K_n}|} \xrightarrow{n \rightarrow \infty} 0.$$

Thus, G_{K_n} are a tight sequence of graphs. Finally, the conditions in Definition 2.2 imply the limit (4). Thus, by Proposition 5.5 the spanning tree entropies for H_n and G_{K_n} are equal.

Let $G_n^b = G_{\mathcal{L}}^b / (n\Lambda)$, which is a balanced bipartite toroidal graph. Recall $e(G_K) = c(K)$. By the results above, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log \det(K_n)}{c(K_n)} &= \lim_{n \rightarrow \infty} \frac{\log \tau(G_{K_n})}{e(G_{K_n})} \\ &= \lim_{n \rightarrow \infty} \frac{|G_{K_n}|}{e(G_{K_n})} \cdot \frac{\log \tau(G_{K_n})}{|G_{K_n}|} \\ &= \frac{|G_L|}{e(G_L)} \lim_{n \rightarrow \infty} \frac{\log \tau(G_{K_n})}{|G_{K_n}|} \quad (\text{by Lemma 5.2}) \\ &= \frac{|G_L|}{e(G_L)} \lim_{n \rightarrow \infty} \frac{\log \tau(H_n)}{|H_n|} \quad (\text{by Proposition 5.5}) \\ &= \frac{1}{e(G_L)} \lim_{n \rightarrow \infty} \frac{\log \tau(H_n)}{n^2} \cdot \frac{n^2 |G_L|}{|H_n|} \\ &= \frac{1}{e(G_L)} \lim_{n \rightarrow \infty} \frac{\log \tau(H_n)}{n^2} \quad (\text{by equation (5)}) \\ &= \frac{1}{e(G_L)} \lim_{n \rightarrow \infty} \frac{\log Z(G_n^b)}{n^2} \quad (\text{by Proposition 5.3}) \\ &= \frac{m(p(z, w))}{c(L)} \quad (\text{by Theorem 3.1}). \end{aligned}$$

□

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